LUSZTIG'S CANONICAL QUOTIENT AND GENERALIZED DUALITY

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Abstract. We give a new characterization of Lusztig's canonical quotient, a finite group attached to each special nilpotent orbit of a complex semisimple Lie algebra. This group plays an important role in the classification of unipotent representations of finite groups of Lie type.

We also define a duality map. To each pair of a nilpotent orbit and a conjugacy class in its fundamental group, the map assigns a nilpotent orbit in the Langlands dual Lie algebra. This map is surjective and is related to a map introduced by Lusztig (and studied by Spaltenstein). When the conjugacy class is trivial, our duality map is just the one studied by Spaltenstein and by Barbasch and Vogan which has image consisting of the special nilpotent orbits.

1. Introduction

To each special nilpotent orbit in a simple Lie algebra over the complex numbers, Lusztig has assigned a finite group (called Lusztig's canonical quotient) which is naturally a quotient of the fundamental group of the orbit. This group is defined using the Springer correspondence and the generic degrees of Hecke algebra representations; it plays an important role in Lusztig's work on parametrizing unipotent representations of a finite group of Lie type [L2].

In this paper we give a description of the canonical quotient which uses the description of the component group of a nilpotent element given in [So2] when G is of adjoint type. To do this we assign to each conjugacy class in the component group a numerical value (called the $\tilde{b}$-value). The definition of the $\tilde{b}$-value makes use of the duality map of [Sp] and involves the dimension of a Springer fiber; it is related to the usual b-value of a Springer representation (the smallest degree in which the representation appears in the harmonic polynomials on the Cartan subalgebra). Our observation is that the kernel of the quotient map from the component group to the canonical quotient consists of all conjugacy classes whose $\tilde{b}$-value is equal to the $\tilde{b}$-value of the trivial conjugacy class.

In the second part of the paper we define a surjective map from the set of pairs consisting of a nilpotent orbit and a conjugacy class in its fundamental group to the set of nilpotent orbits in the Langlands dual Lie algebra. This extends the duality of [Sp] (which corresponds to the situation when the conjugacy class is trivial). In fact the map depends only on the image of the conjugacy class in Lusztig's canonical quotient. Our map is defined by combining the duality of [Sp] and a map defined by Lusztig [L2]. Lusztig's map can be thought of as a generalization of induction [LuSp]; in his book, Spaltenstein also studied this generalization of induction [Sp].

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2. Basic set-up

Throughout the paper, G is a connected simple algebraic group over the complex numbers $\mathbf{C}$. In G we fix a maximal torus T contained in a Borel subgroup B. Let the character group of T be $X^*(T)$. We use $\mathfrak{g}$ for the Lie algebra of G. Let Φ be the roots of G and let $W = N_G(T)/T$ be the Weyl group of G (with respect to T).

The group G acts on $\mathfrak{g}$ via the adjoint action. If $e \in \mathfrak{g}$ is an element of $\mathfrak{g}$, we denote by $\mathcal{O}_e$ the orbit of $\mathfrak{g}$ through e under the action of G. If e is a nilpotent element, we call $\mathcal{O}_e$ a nilpotent orbit.

We denote by $\mathcal{N}_o$ the set of nilpotent orbits in $\mathfrak{g}$. This set is partially ordered by the relation $\mathcal{O}_e \preceq \mathcal{O}_f$ whenever $\bar{\mathcal{O}}_e \subset \bar{\mathcal{O}}_f$.

For $\mathcal{O} \in \mathcal{N}_o$ and $e \in \mathcal{O}$, let $A(e)$ be the component group $Z_G(e)/Z_G^0(e)$. If $e, e' \in \mathcal{O}$, then $A(e)$ may be identified with $A(e')$ and so we write $A(\mathcal{O})$ for this finite group. Furthermore, we may speak in a well-defined manner of the conjugacy classes of $A(\mathcal{O})$. When G is simply-connected (and we pass to the analytic topology), $A(\mathcal{O})$ is just the fundamental group of $\mathcal{O}$. We denote by $\hat{A}(\mathcal{O})$ the irreducible representations of $A(\mathcal{O})$ and use the notion of irreducible local system on $\mathcal{O}$ interchangeably with the notion of irreducible representation of $A(\mathcal{O})$, or $A(e)$ for any $e \in \mathcal{O}$.

We denote by $\mathcal{N}_{o,c}$ the set of pairs $(\mathcal{O}, C)$ consisting of an orbit $\mathcal{O} \in \mathcal{N}_o$ and a conjugacy class $C \subset A(\mathcal{O})$. The set of special orbits in $\mathcal{N}_o$ (see [L1]) will be denoted by $\mathcal{N}_o^{sp}$. There is an order-reversing duality map $d : \mathcal{N}_o \to \mathcal{N}_o^{sp}$ studied by Spaltenstein such that d^2 is the identity on $\mathcal{N}_o^{sp}$ [Sp]. This map is already implicit in [L1], so we refer to d henceforth as Lusztig-Spaltenstein duality.

3. Notation on partitions

In the classical groups it will be helpful to have a description of the elements of $\mathcal{N}_o$ and $\mathcal{N}_{o,c}$ and the map d in terms of partitions. We introduce that notation now (roughly following the references [CM], [Ca], and [Sp]).

Let $\mathcal{P}(m)$ denote the set of partitions $\lambda = [\lambda_1 \geq \cdots \geq \lambda_k]$ of m (we assume that $\lambda_k \neq 0$). For a part λ_i of λ, we call i the position of λ_i in λ. For $\lambda \in \mathcal{P}(m)$, let $r_i = \#\{j \mid \lambda_j = i\}$. For $\epsilon \in \{0, 1\}$ let $\mathcal{P}_\epsilon(m)$ be the set of partitions λ of m such that $r_i \equiv 0$ whenever $i \equiv \epsilon$ (all congruences are modulo 2).

It is well known that $\mathcal{N}_o$ is in bijection with $\mathcal{P}(n+1)$ when $\mathfrak{g}$ is of type A_n; with $\mathcal{P}_1(2n)$ when $\mathfrak{g}$ is of type C_n; with $\mathcal{P}_0(2n+1)$ when $\mathfrak{g}$ is of type B_n; and with $\mathcal{P}_0(2n)$ when $\mathfrak{g}$ is of type D_n, except that those partitions with all even parts correspond to two orbits in $\mathcal{N}_o$ (called the very even orbits). In what follows we will never have a need to address separately the very even orbits, so we will not bother to introduce notation to distinguish between very even orbits.

We will also refer to $\mathcal{P}_1(2n)$ as $\mathcal{P}_C(2n)$; to $\mathcal{P}_0(2n+1)$ as $\mathcal{P}_B(2n+1)$; and to $\mathcal{P}_0(2n)$ as $\mathcal{P}_D(2n)$.

The set $\mathcal{P}(m)$ is partially ordered by the usual partial ordering on partitions. This induces a partial ordering on the sets $\mathcal{P}_C(2n)$, $\mathcal{P}_B(2n+1)$, and $\mathcal{P}_D(2n)$ and these partial orderings coincide with the partial ordering on nilpotent orbits given by inclusion of closures. We will refer to nilpotent orbits and partitions interchangeable in the classical groups (with the caveat mentioned earlier for the very even orbits in type D).

We can also represent elements of $\mathcal{N}_{o,c}$ in terms of partitions in the classical groups (see the last section of [So2]). Given $(\mathcal{O}, C) \in \mathcal{N}_{o,c}$ let $e \in \mathcal{O}$ and let $s \in Z_G(e)$ be a semisimple element whose image in $A(e) \cong A(\mathcal{O})$ belongs to C. Set $\mathfrak{l} = Z_{\mathfrak{g}}(s)$. We may as well assume that $\mathfrak{l}$ contains Lie(T); then the subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ (which we called a pseudo-Levi subalgebra in [So2]) corresponds to a proper subset of the extended Dynkin diagram of $\mathfrak{g}$. It is always possible to choose s so that $\mathfrak{l}$ has semisimple rank equal to $\mathfrak{g}$, which we will do. Next we write $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$, where $\mathfrak{l}_2$ is a semisimple subalgebra of the same type as $\mathfrak{g}$ and $\mathfrak{l}_1$ is a simple Lie algebra (or possibly zero) which contains the root space corresponding to the lowest root of $\mathfrak{g}$ (the extra node in the extended Dynkin diagram). Then $\mathfrak{l}_1$ (if non-zero) is of type D, C, or D depending on whether $\mathfrak{g}$ is of type B, C, or D, respectively. Write $e = e_1 + e_2$ where $e_1 \in \mathfrak{l}_1$ and $e_2 \in \mathfrak{l}_2$. Finally, we may modify the choice of s so that e_1 is a distinguished nilpotent element in $\mathfrak{l}_1$. Then we can attach to $(\mathcal{O}, C) \in \mathcal{N}_{o,c}$ a pair of partitions (ν, η) where ν is the partition of e_1 in $\mathfrak{l}_1$ and η is the partition of e_2 in $\mathfrak{l}_2$. We have

$$\begin{aligned} &\nu \in \mathcal{P}_D(2k),\ \eta \in \mathcal{P}_B(2n+1-2k) && \text{in type } B_n \\ &\nu \in \mathcal{P}_C(2k),\ \eta \in \mathcal{P}_C(2n-2k) && \text{in type } C_n \\ &\nu \in \mathcal{P}_D(2k),\ \eta \in \mathcal{P}_D(2n-2k) && \text{in type } D_n \end{aligned} \tag{1}$$

where $0 \leq k \leq n$ and ν is a distinguished partition, meaning $r_i \leq 1$ for all i (this necessarily forces $r_i = 0$ whenever $i \equiv \epsilon$). The partition λ of $\mathcal{O}$ is just the partition consisting of all the parts in ν and η. We denote this partition by $\nu \cup \eta$ and write $\lambda = \nu \cup \eta$. In types C and D we make the further assumption that the largest $i \not\equiv \epsilon$ with $r_i \equiv 1$ is not a part of ν. These assumptions guarantee that when G is of adjoint type there is a bijection between $\mathcal{N}_{o,c}$ and the pairs (ν, η) specified above (if $\mathcal{O}$ is very even in type D, the component group $A(\mathcal{O})$ is trivial when G is adjoint; so there are no additional complications beyond the one mentioned earlier). We note that ν will be the empty partition (that is, $k = 0$ in equation (1)) if and only if C is the trivial conjugacy class.

For example, let $\mathcal{O}$ be the orbit with partition $[5, 3, 1]$ in B_4. Then we have

$$(\emptyset, [5, 3, 1]), ([3, 1], [5]), ([5, 1], [3]), ([5, 3], [1])$$

are the four elements of $\mathcal{N}_{o,c}$ corresponding to the four conjugacy classes of $A(\mathcal{O})$ when G is of adjoint type, with the first pair corresponding to the trivial conjugacy class.

The dual partition of $\lambda \in \mathcal{P}(m)$, denoted λ^*, is defined by $\lambda^*_i = \#\{j \mid \lambda_j \geq i\}$. Let $X = B$, C, or D. We define the X-collapse of a partition $\lambda \in \mathcal{P}(m)$ where m is even when $X = C$ or D and m is odd when $X = B$. The X-collapse of $\lambda \in \mathcal{P}(m)$ is the partition $\mu \in \mathcal{P}_X(m)$ such that $\mu \preceq \lambda$ and such that $\mu' \preceq \mu$ whenever $\mu' \in \mathcal{P}_X(m)$ and $\mu' \preceq \lambda$. The X-collapse of λ is denoted λ_X. It is well-defined and unique.

The duality map $d : \mathcal{N}_o \to \mathcal{N}_o^{sp}$ can now be expressed as follows in the classical groups. In type A, we have $d(\lambda) = \lambda^*$ and in type X (for $X = B, C, D$) we have $d(\lambda) = (\lambda^*)_X$. The set of special nilpotent orbits correspond to the partitions in the image of d (in type D all

very even orbits are special). Hence in type A all nilpotent orbits are special. In the other types it is known that λ is special if and only if

$$\begin{aligned} &\lambda^* \in \mathcal{P}_B(2n+1) \text{ in type } B_n \\ &\lambda^* \in \mathcal{P}_C(2n) \text{ in type } C_n \\ &\lambda^* \in \mathcal{P}_C(2n) \text{ in type } D_n. \end{aligned} \tag{2}$$

4. $\tilde{b}$-value

In this section we assign to each pair $(\mathcal{O}, C) \in \mathcal{N}_{o,c}$ a natural number which we will call the $\tilde{b}$-value of $(\mathcal{O}, C)$ and which will be denoted by $\tilde{b}_{(\mathcal{O},C)}$.

Given $(\mathcal{O}, C) \in \mathcal{N}_{o,c}$ let $e \in \mathcal{O}$ and $s \in Z_G(e)$ be a semisimple element whose image in $A(e) \cong A(\mathcal{O})$ belongs to C. The group $L = Z_G(s)$ has as its Lie algebra $\mathfrak{l} = Z_{\mathfrak{g}}(s)$ and clearly $e \in \mathfrak{l}$.

Let $\mathcal{O}_e^{\mathfrak{l}}$ denote the orbit in $\mathfrak{l}$ through e under the action of L. Applying Lusztig-Spaltenstein duality $d_{\mathfrak{l}}$ with respect to $\mathfrak{l}$, we obtain the orbit $\mathcal{O}_f^{\mathfrak{l}} = d_{\mathfrak{l}}(\mathcal{O}_e^{\mathfrak{l}}) \subset \mathfrak{l}$. We define $\tilde{b}_{(\mathcal{O},C)} = \frac{1}{2}(\dim(\mathfrak{l}) - \dim(\mathcal{O}_f^{\mathfrak{l}}) - \dim(T))$. Equivalently, if $\mathcal{B}_f^{\mathfrak{l}}$ is the variety of Borel subalgebras of $\mathfrak{l}$ which contain f, then $\tilde{b}_{(\mathcal{O},C)} = \dim \mathcal{B}_f^{\mathfrak{l}}$. We have the following proposition whose proof we give after the proof of proposition 7.

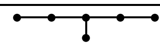
Proposition 1. *The number $\tilde{b}_{(\mathcal{O},C)}$ is well-defined. In other words, it is independent of the choices made for s and e.*

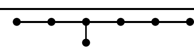
By results in [So2] it is always possible to choose s so that e is distinguished in $\mathfrak{l} = Z_{\mathfrak{g}}(s)$. By the previous proposition, in order to compute $\tilde{b}_{(\mathcal{O},C)}$ it is enough to determine a pair $(\mathfrak{l}, e)$ attached to $(\mathcal{O}, C)$ where e is distinguished in $\mathfrak{l}$ (this is done in [So2]) and to compute the $\tilde{b}$-value of e with respect to $\mathfrak{l}$ (with the trivial conjugacy class 1 of $A(e)$). Hence it suffices to compute $\tilde{b}_{(\mathcal{O},1)}$ for each distinguished orbit $\mathcal{O}$ in each simple Lie algebra.

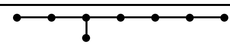
We now record $\tilde{b}_{(\mathcal{O},1)}$ for the distinguished orbits in the exceptional groups.

G_2		
	Bala-Carter	$\tilde{b}$
2 2	G_2	6
2 0	$G_2(a_1)$	1

F_4		
	Bala-Carter	$\tilde{b}$
2 2 2 2	F_4	24
2 2 0 2	$F_4(a_1)$	13
0 2 0 2	$F_4(a_2)$	10
0 2 0 0	$F_4(a_3)$	4

E_6		
	Bala-Carter	$\tilde{b}$
2 2 2 2 2 2	E_6	36
2 2 0 2 2 2	$E_6(a_1)$	25
2 0 2 0 2 0	$E_6(a_3)$	15

E_7		
	Bala-Carter	$\tilde{b}$
2 2 2 2 2 2 2	E_7	63
2 2 0 2 2 2 2	$E_7(a_1)$	46
2 2 0 2 0 2 2	$E_7(a_2)$	37
2 0 2 0 2 2 0	$E_7(a_3)$	30
2 0 2 0 0 2 0	$E_7(a_4)$	22
0 0 2 0 0 2 0	$E_7(a_5)$	16

E_8		
	Bala-Carter	$\tilde{b}$
2 2 2 2 2 2 2 2	E_8	120
2 2 0 2 2 2 2 2	$E_8(a_1)$	91
2 2 0 2 0 2 2 2	$E_8(a_2)$	74
2 0 2 0 2 2 2 0	$E_8(a_3)$	63
2 0 2 0 2 0 2 0	$E_8(a_4)$	52
2 0 2 0 0 2 2 0	$E_8(b_4)$	47
2 0 2 0 0 2 0 0	$E_8(a_5)$	42
0 0 2 0 0 2 2 0	$E_8(b_5)$	37
0 0 2 0 0 2 0 0	$E_8(a_6)$	32
0 0 2 0 0 0 2 0	$E_8(b_6)$	28
0 0 0 2 0 0 0 0	$E_8(a_7)$	16

We now record $\tilde{b}_{(\mathcal{O},1)}$ for the distinguished orbits in the classical groups. In fact, it is no harder to record the formula for $\tilde{b}_{(\mathcal{O},1)}$ for any orbit $\mathcal{O}$, whether distinguished or not. Let $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k]$ be the partition for $\mathcal{O}$ in the appropriate classical group. Then $\tilde{b}_{(\mathcal{O},1)}$ equals

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - \frac{1}{2}\sum_{i=1}^{k}\lambda_i \text{ in type } A \\ \frac{1}{4}\sum_{i=1}^{k}\lambda_i^2 - \frac{1}{2}\sum_{i \text{ odd}}\lambda_i + \frac{1}{4} \text{ in type } B \\ \frac{1}{4}\sum_{i=1}^{k}\lambda_i^2 - \frac{1}{2}\sum_{i \text{ even}}\lambda_i \text{ in types } C \text{ and } D. \end{aligned} \tag{3}$$

We omit the calculations (which are easy in all types except type D) since they will follow quickly from later work.

Remark 2. If $\mathcal{O}$ is the Richardson orbit for the parabolic subgroup P whose Levi subgroup has root system isomorphic to $A_1 \times \cdots \times A_1$ (s-times), we have observed that $\tilde{b}_{(\mathcal{O},1)} = N - s\ m_{n+1-s}$. Here, n is the rank of G, N is the number of positive roots of G, and m_i is the i-th exponent of G when the exponents are listed in increasing order. This offers the hope that there is a way to determine the $\tilde{b}$-value of any distinguished orbit from standard data arising purely from the root system.

The next proposition is analogous to a statement (due to Lusztig in [L2]) about a-values of Springer representations which we will recall in the next section.

Proposition 3. *Let* 1 *be the trivial conjugacy and let* C *be any conjugacy class in* $A(\mathcal{O})$*. Then* $\tilde{b}_{(\mathcal{O},1)} \leq \tilde{b}_{(\mathcal{O},C)}$.

Proof. In the exceptional groups, we have verified this directly. In the classical groups, we must show that for any $\lambda \in \mathcal{N}_o$ and $(\nu, \eta) \in \mathcal{N}_{o,c}$ with $\lambda = \nu \cup \eta$ that we have $\tilde{b}_\lambda \leq \tilde{b}_{(\nu,\eta)} = \tilde{b}_\nu + \tilde{b}_\eta$ where the $\tilde{b}$-value is computed with respect to the appropriate subalgebras for λ, ν, and η.

In type B, $\tilde{b}_\lambda = \frac{1}{4}\sum_{i=1}^k \lambda_i^2 - \frac{1}{2}\sum_{i \text{ odd}} \lambda_i + \frac{1}{4}$. For any (ν, η) with $\lambda = \nu \cup \eta$, we must use both the $\tilde{b}$-value formula in type B for η and in type D for ν. It is clear that $\tilde{b}_\nu + \tilde{b}_\eta = \frac{1}{4}\sum_{i=1}^k \lambda_i^2 + \frac{1}{4}$ minus various λ_i. It turns out that that for each m either λ_{2m-1} or λ_{2m} will be subtracted in the formula, but not both, as we now show. There are four possibilities for each m. If both λ_{2m-1} and λ_{2m} belong to η or both belong to ν, then $\tilde{b}_{(\nu,\eta)}$ will have a term $-\lambda_{2m-1}$ or $-\lambda_{2m}$, but not both, since the formulas for B and D select every other λ_i to subtract. If on the other hand, λ_{2m-1} belongs to η and λ_{2m} belongs to ν, then again $\tilde{b}_{(\nu,\eta)}$ will have a term $-\lambda_{2m-1}$ or $-\lambda_{2m}$, but not both. This is because the parity of the position of λ_{2m-1} in η will be the same as the parity of the position of λ_{2m} in ν (as the position of λ_{2m} in λ is an even number, namely $2m$). But the formula for B and D choose to subtract parts whose positions have opposite parity. Hence, only one can be selected in the formula for $\tilde{b}_\nu + \tilde{b}_\eta$. The result is the same if λ_{2m-1} belongs to ν and λ_{2m} belongs to η. Thus in either of the four cases, the effect is to subtract a number which is less than or equal to λ_{2m-1} since $\lambda_{2m-1} \geq \lambda_{2m}$, hence the inequality $\tilde{b}_\lambda \leq \tilde{b}_\nu + \tilde{b}_\eta$.

In types C and D, a similar argument holds, except we look at the consecutive parts λ_{2m} and λ_{2m+1} of λ (the part λ_1 will never be subtracted in the formulas for the $\tilde{b}$-value). One or the other, but not both, of these parts will be subtracted in the formula for $\tilde{b}_\nu + \tilde{b}_\eta$ demonstrating the inequality. □

We recall some notation from [So2] in the classical groups

$$S_{odd} = \{i \in \mathbb{N} |\ i \not\equiv \varepsilon,\ r_i \equiv 1\}$$
$$S_{even} = \{i \in \mathbb{N} |\ i \not\equiv \varepsilon,\ r_i \equiv 0 \text{ and } r_i \neq 0\}.$$

List the elements of S_{odd} in decreasing order $j_l \geq j_{l-1} \geq \cdots \geq j_2 \geq j_1$. Assume that l is even in type C by setting $j_1 = 0$ if necessary (l is automatically odd in type B and automatically even in type D).

An element $(\nu, \eta) \in \mathcal{N}_{o,c}$ determines two sets T_1 and T_2 such that $T_1 \subset S_{odd}$ and $T_2 \subset S_{even}$ coming from the parts of ν. Namely, the parts of ν (which each occur with multiplicity one since ν is distinguished) consist precisely of the elements in $T_1 \cup T_2$. For $j \in S_{odd}$, let

$\delta_j = 1$ if $j \in T_1$ and 0 if $j \notin T_1$. Note we are assuming that $\delta_l = 0$ in types C and D. We define subsets $T_2^{(m)}$ of T_2 as follows: let $T_2^{(m)}$ consist of those $i \in T_2$ such that $j_{m+1} \geq i \geq j_m$ and define t_m to be the cardinality of $T_2^{(m)}$.

The next proposition follows easily from the previous proposition and the formulas for the $\tilde{b}$-value.

Proposition 4. *In the classical groups (not of type A), the equality $\tilde{b}_{(\mathcal{O},1)} = \tilde{b}_{(\mathcal{O},C)}$ holds if and only if $t_m = 0$ when m is even and $\delta_{m+1} + t_m + \delta_m$ is even when m is odd.*

5. Lusztig's canonical quotient

Our first main result is a new description of Lusztig's canonical quotient $\bar{A}(\mathcal{O})$, which is a quotient of $A(\mathcal{O})$. These finite groups play an important role in Lusztig's classification of unipotent representations of finite groups of Lie type. Namely, the set of unipotent representations for G (split, with connected center, over a finite field) is parametrized by the following data: a special nilpotent orbit $\mathcal{O}$, an element $x \in \bar{A}(\mathcal{O})$, and an irreducible representation of the centralizer of x in $\bar{A}(\mathcal{O})$ (all up to the appropriate conjugation). We hope to obtain a better understanding of this parametrization by having such an explicit description of the canonical quotient and its conjugacy classes.

Let us recall Lusztig's definition of $\bar{A}(\mathcal{O})$. Although Lusztig assumed $\mathcal{O}$ is special, his definition remains valid even if $\mathcal{O}$ is not special, so in what follows we do not assume $\mathcal{O}$ is special.

Recall that for each nilpotent element e and local system π on $\mathcal{O}_e$, Springer has defined a representation $E_{e,\pi}$ which (if non-zero) is an irreducible representation of W. Recall also that each irreducible representation E of W comes with two important numerical invariants. One comes from the generic degree of E (the a-value) and one comes from the fake degree of E (the b-value). We refer to [L2] for the definitions. Note that our notation is consistent with [L2], but is not consistent with [L1] or [Ca]. In those sources, our b-value is their a-value and our a-value is their $\tilde{a}$-value. The original definition of the canonical quotient of $A(\mathcal{O})$ is as follows. Given $e \in \mathcal{O}$, consider the set

$$\mathcal{S} = \{\pi \in \hat{A}(e) \mid E_{e,\pi} \neq 0 \text{ and } a_{E_{e,\pi}} = a_{E_{e,1}}\}$$

where $\hat{A}(e)$ is the set of irreducible representations of $A(e)$ and $E_{e,1}$ denotes the Springer representation associated to the trivial representation of $A(e)$. Let H be the intersection of all the kernels of the representations in $\mathcal{S}$. Then $\bar{A}(\mathcal{O})$ is defined to be the quotient $A(\mathcal{O})/H$ [L2]. If a local system π is not equivariant for G when G is of adjoint type, then $E_{e,\pi}$ will be zero. Hence, the canonical quotient is the same for groups in the same isogeny class of G and so we assume G is of adjoint type in what follows.

Remark 5. Lusztig has observed that for $\pi \in \hat{A}(e)$ and $E_{e,\pi} \neq 0$ that $a_{E_{e,\pi}} \leq a_{E_{e,1}}$. Compare this with proposition 3.

In light of Lusztig's original definition and proposition 3 we consider all conjugacy classes C in $A(\mathcal{O})$ with the property that $\tilde{b}_{(\mathcal{O},C)} = \tilde{b}_{(\mathcal{O},1)}$. Let H' be the union of all such conjugacy classes.

Theorem 6. *The set H' coincides with H, so that $\bar{A}(\mathcal{O}) = A(\mathcal{O})/H'$. Moreover, the $\tilde{b}$-value is constant on the cosets of $H' = H$ in $A(\mathcal{O})$ (which are always a union of conjugacy classes).*

Proof. We are assuming that G is of adjoint type (the result is still valid for any G). In the exceptional groups we verified the results directly using the tables for a-values and Springer representations in [Ca], the tables of conjugacy classes in $A(\mathcal{O})$ in [So2], and knowledge of the $\tilde{b}$-values for distinguished orbits given above.

In the classical groups we need to do some work in order to understand which local systems appear in the set $\mathcal{S}$.

We illustrate the situation in type B_n. Let $\mathcal{O}$ have partition $\lambda = [\lambda_1 \geq \cdots \geq \lambda_k]$. As before, r_i is the number of parts of λ of size i. So r_i is even whenever i is even as we are in type B. We associate to $\mathcal{O}$ a symbol as in chapter 13.3 of [Ca]. It consists of k elements. Let $d_i = \sum_{j<i} r_j$. Each odd i contributes to the symbol the interval of length r_i

$$(d_i + \frac{i-1}{2}, d_i + \frac{i-1}{2} + 1, \ldots, d_i + \frac{i-1}{2} + r_i - 1)$$

and each even i contributes to the symbol the $r_i/2$ numbers

$$d_i + i/2, d_i + i/2 + 2, \ldots, d_i + i/2 + r_i - 2$$

each repeated twice in weakly increasing order.

Consider the elementary 2-group with basis consisting of elements x_i, one for each odd i with $r_i \neq 0$. Then $A(\mathcal{O})$ is the subgroup of this group consisting of elements expressible as a sum of an even number of basis elements (this is because we are working in the special orthogonal group and not in the full orthogonal group). Representations π of $\hat{A}(\mathcal{O})$ are thus specified by their values (of ± 1) on the x_i's. The representations π for which $E_{e,\pi}$ is non-zero are those with the property that

$$\#\{x_i \mid r_i \equiv 1, d_i \equiv 0 \text{ and } \pi(x_i) = -1\} = \#\{x_i \mid r_i \equiv 1, d_i \equiv 1 \text{ and } \pi(x_i) = -1\}.$$

If r_i is even, π may have the value 1 or -1 on x_i.

To determine the set $\mathcal{S}$ it is necessary to compute the a-value of each non-zero representation $E_{e,\pi}$ (see chapter 11.4 of [Ca]). We have to convert between two different notations for symbols. This is a bit of a pain, but the work is greatly simplified since we are only interested in when the a-value of $E_{e,\pi}$ equals the a-value of $E_{e,1}$. We find that the set $\mathcal{S}$ consists of the following π. Above we listed those odd i with r_i odd (which was denoted S_{odd} above) in decreasing order as $j_l \geq j_{l-1} \geq \cdots \geq j_1$ (note that l must be odd). If $\pi \in \mathcal{S}$, we must have $\pi(x_{j_l}) = 1$ and $\pi(x_{j_{2m}}) = \pi(x_{j_{2m-1}})$. Let i be odd and r_i even. If $j_{2m+1} \geq i \geq j_{2m}$ for some m, then π may take values ± 1 on x_i. If, on the other hand, $j_{2m} \geq i \geq j_{2m-1}$, then $\pi(x_i) = \pi(x_{j_{2m-1}})$.

Since $A(\mathcal{O})$ is abelian, each element forms its own conjugacy class. We need to relate our two descriptions of conjugacy classes in $A(\mathcal{O})$. Given $x \in A(\mathcal{O})$ write $x = x_{i_1} + x_{i_2} + \cdots + x_{i_{2m}}$ where $i_1 \geq i_2 \geq \cdots \geq i_{2m}$ (the usual classical description). Then set $\nu = [i_1 \geq \cdots \geq i_{2m}]$ and define η by $\lambda = \nu \cup \eta$. Then $(\nu, \eta) \in \mathcal{N}_{o,c}$ exactly corresponds to the conjugacy class of $x \in A(\mathcal{O})$ (see [So1]). Using our previous notation of $T_2^{(m)}$, t_m, and δ_j, we see that H consists of those conjugacy classes where $t_m = 0$ when m is even and $\delta_{m+1} + t_m + \delta_m$ is even when m is odd. By proposition 4 this is exactly the condition that the conjugacy class

(ν, η) belongs to H', showing that $H' = H$. Finally, the statement that $\tilde{b}$ is constant on the cosets on $H' = H$ is an easy computation similar to the one done to find which classes belonged to H'.

A similar proof holds in type C and D which we omit. $\square$

For completeness we record $\bar{A}(\mathcal{O})$ in the classical groups. In the exceptional groups, $\bar{A}(\mathcal{O})$ is listed in the last section.

In type A, $\bar{A}(\mathcal{O})$ is trivial. In the other classical types, $\bar{A}(\mathcal{O})$ is an elementary 2-group. Assuming G is the special orthogonal group in types B and D and G is the symplectic group in type C, we describe a subgroup K of $A(\mathcal{O})$ which maps bijectively onto $\bar{A}(\mathcal{O}) = A(\mathcal{O})/H$.

In types B and D, consider the subgroup K of $A(\mathcal{O})$ consisting of all elements expressible as a sum of an even number of x_i where i is equal to some j_m for m odd or $i \in T_2^{(m)}$ for m even. In type C, consider the subgroup K of $A(\mathcal{O})$ consisting of all elements expressible as a sum of x_i where i is equal to some j_m for m odd or $i \in T_2^{(m)}$ for m even. Then $K \cong \bar{A}(\mathcal{O})$. In other words, the i in question correspond, in type B, to corners of the Young diagram of λ which have odd length and odd height (i is odd and l is odd); in type D, to corners of the Young diagram of λ which have odd length and even height (i is odd and l is even); in type C, to corners of the Young diagram of λ which have even length and even height (i is even and l is even).

6. Duality

Let ${}^L G$ be the Langlands dual group of G with Lie algebra ${}^L\mathfrak{g}$ (in other words, the root data of ${}^L G$ and G are dual). Denote by ${}^L\mathcal{N}_o$ the set of nilpotent orbits in ${}^L\mathfrak{g}$ and ${}^L\mathcal{N}_o^{sp}$ the set of special nilpotent orbits in ${}^L\mathfrak{g}$.

There is a natural order-preserving (and dimension-preserving) bijection between $\mathcal{N}_o^{sp}$ and ${}^L\mathcal{N}_o^{sp}$. Therefore we can also view Lusztig-Spaltenstein duality as a map $d : \mathcal{N}_o \to {}^L\mathcal{N}_o^{sp}$. It is in this context that Barbasch and Vogan have given a representation-theoretic description of Lusztig-Spaltenstein duality using primitive ideals [BV]. To distinguish between the duality which stays within G and the version which passes to the dual group, we use the notation $d(\mathcal{O})$ to mean the dual orbit within $\mathfrak{g}$ and we use the notation $d_{(\mathcal{O},1)}$ to mean the dual orbit within ${}^L\mathfrak{g}$. This is consistent with later (and earlier) notation.

We will now define a map $d : \mathcal{N}_{o,c} \to {}^L\mathcal{N}_o$ with the property that it extends Lusztig-Spaltenstein duality. It will turn out that d is surjective and depends only on the image of C in the canonical quotient $\bar{A}(\mathcal{O})$. Moreover, if $e' \in d_{(\mathcal{O},C)}$, then we will have $\dim({}^L\mathcal{B}_{e'}) = \tilde{b}_{(\mathcal{O},C)}$ where ${}^L\mathcal{B}$ denotes the flag variety of ${}^L G$.

The definition of d is as follows. Given $(\mathcal{O}, C) \in \mathcal{N}_{o,c}$, pick $e \in \mathcal{O}$ and a semisimple element $s \in Z_G(e)$ whose image in $A(e) \cong A(\mathcal{O})$ belongs to C. The group $L = Z_G(s)$ has as its Lie algebra $\mathfrak{l} = Z_{\mathfrak{g}}(s)$ and clearly $e \in \mathfrak{l}$. Let $\mathcal{O}_e^{\mathfrak{l}}$ denote the orbit in $\mathfrak{l}$ through e under the action of L. Applying Lusztig-Spaltenstein duality $d_{\mathfrak{l}}$ with respect to $\mathfrak{l}$ (we assume here that $d_{\mathfrak{l}}$ stays within $\mathfrak{l}$ and we don't pass to the dual group), we obtain the orbit $\mathcal{O}_f^{\mathfrak{l}} = d_{\mathfrak{l}}(\mathcal{O}_e^{\mathfrak{l}}) \subset \mathfrak{l}$. At this point we simply apply Lusztig's map from Chapter 13 of [L2].

But first we recall the following result due to Joseph and proved uniformly in [BM]. Every Weyl group representation of the form $E_{e,1}$ possesses property (B) of [LuSp] with respect to G. That is, the b-value of $E_{e,1}$ coincides with $\dim(\mathcal{B}_e)$ and moreover the multiplicity of

$E_{e,1}$ in the harmonic polynomials of degree $\dim(\mathcal{B}_e)$ on a Cartan subalgebra of $\mathfrak{g}$ is exactly one.

The Springer correspondence for L produces an irreducible representation $E_{f,1}$ (always non-zero) of $W(s)$, the Weyl group of $\mathfrak{l} = Z_{\mathfrak{g}}(s)$. The second part of the statement of property (B) means that we can apply the operation of truncated induction to $E_{f,1}$ (see [L1]). So let $E = j_{W(s)}^{W}(E_{f,1})$ be the representation of W obtained by truncated induction of $E_{f,1}$ from $W(s)$ to W. Then E has the property that $b_E = b_{E_{f,1}}$ from the definition of truncated induction. Now consider E as a representation of ${}^{L}W$, the Weyl group of ${}^{L}G$ (since ${}^{L}W$ is isomorphic to W via the involution which interchanges long and short roots). Lusztig has observed that E is always of the form $E_{e',1}$ for some nilpotent element e' in ${}^{L}\mathfrak{g}$ (this will be re-verified explicitly below). We then define $d_{(\mathcal{O},C)}$ to be the orbit in ${}^{L}\mathfrak{g}$ through e'.

It is clear from the definition (assuming it is well-defined) and the first part of the statement of property (B) (applied twice, once in L and once in ${}^{L}G$) that $\dim({}^{L}\mathcal{B}_{e'}) = \tilde{b}_{(\mathcal{O},C)}$ as promised above.

Proposition 7. *The duality map is well-defined; that is, it is independent of the choices made for s and e.*

Proof. We show that the representation E constructed above is independent of s (it is clearly independent of e, since Springer representations only depend on the orbit through e).

One of the main properties of Lusztig-Spaltenstein duality is that if $e \in \mathfrak{l}'$ where $\mathfrak{l}'$ is a Levi subalgebra of $\mathfrak{g}$, then

$$d_{\mathfrak{g}}(\mathcal{O}_e^{\mathfrak{g}}) = \mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{g}} d_{\mathfrak{l}'}(\mathcal{O}_e^{\mathfrak{l}'}).$$

The notation on the right-side is Lusztig-Spaltenstein induction. According to [LuSp] and the validity of property (B), one therefore has that

$$E_{d_{\mathfrak{g}}(\mathcal{O}_e),1} = j_{W(\mathfrak{l}')}^{W}(E_{d_{\mathfrak{l}}(\mathcal{O}_e^{\mathfrak{l}}),1})$$

where $W(\mathfrak{l}')$ is the Weyl group of $\mathfrak{l}'$.

Now let S be a maximal torus in $Z_G(s,e)$. Then $\mathfrak{l}' = Z_{\mathfrak{g}}(s,S)$ is a Levi subalgebra of $\mathfrak{l} = Z_{\mathfrak{g}}(s)$ and $e \in \mathfrak{l}' \subset \mathfrak{l}$. Moreover $\mathfrak{l}'$ is of the form $Z_{\mathfrak{g}}(s')$ for a semisimple element $s' \in Z_G(e)$ and the image of s' in $A(e)$ necessarily belongs to C. By the transitivity of truncated induction applied to the sequence $\mathfrak{l}' \subset \mathfrak{l} \subset \mathfrak{g}$, we see that the representation E is the same whether we work with respect to $\mathfrak{l}$ or $\mathfrak{l}'$, i.e. whether we work with s or s'. The main result of [So2] is that the pair $(\mathfrak{l}', e)$ (up to simultaneous conjugation by elements in G) is determined by (and determines in the case when G is of adjoint type) the conjugacy class $C \subset A(\mathcal{O}_e)$. In other words, $(\mathfrak{l}', e)$ (up to simultaneous conjugation by elements in G) is independent of the choice of s and thus the construction of E is independent of the choices made for s and e.

$\square$

Proposition 1 now follows from the above proof either by invoking property (B) in L, or more simply by applying the dimension formula $\dim d_{\mathfrak{l}}(\mathcal{O}_e^{\mathfrak{l}}) = \dim(\mathfrak{l}) - \dim(\mathfrak{l}') + \dim d_{\mathfrak{l}'}(\mathcal{O}_e^{\mathfrak{l}'})$ for induction of orbits from the Levi subalgebra $\mathfrak{l}'$ of $\mathfrak{l}$.

7. Duality in Classical Groups

We now calculate the duality map for the classical groups not of type A (there is nothing new here in type A). First, we assign to a partition λ (which may or may not correspond to a nilpotent orbit in that classical group) a representation E_λ of the Weyl group of that group. When λ corresponds to an actual nilpotent orbit, E_λ is just the Springer representation $E_{\lambda,1}$ for the orbit with trivial local system.

Let $\lambda \in \mathcal{P}(m)$ where m is even in types C, D and odd in type B. Form the dual partition λ^*. Separate the parts of λ^* into its odd parts $2\alpha_1 + 1 \geq 2\alpha_2 + 1 \geq 2\alpha_3 + 1 \cdots \geq 2\alpha_r + 1$ and its even parts $2\beta_1 \geq 2\beta_2 \geq 2\beta_3 \cdots \geq 2\beta_s$. We then associate to λ the representation $E_\lambda = j_{W'}^{W}(\epsilon_{W'})$ where W' is the Weyl group of the subsystem of type

$$\Phi' = \begin{cases} B_{\alpha_1} \times B_{\beta_1} \times D_{\alpha_2+1} \times D_{\beta_2} \times B_{\alpha_3} \times B_{\beta_3} \times D_{\alpha_4+1} \times D_{\beta_4} \dots & \text{in type } B, \\ D_{\alpha_1+1} \times C_{\beta_1} \times C_{\alpha_2} \times D_{\beta_2} \times D_{\alpha_3+1} \times C_{\beta_3} \times C_{\alpha_4} \times D_{\beta_4} \dots & \text{in type } C, \\ D_{\alpha_1+1} \times D_{\beta_1} \times B_{\alpha_2} \times B_{\beta_2} \times D_{\alpha_3+1} \times D_{\beta_3} \times B_{\alpha_4} \times B_{\beta_4} \dots & \text{in type } D. \end{cases} \tag{4}$$

and $\epsilon_{W'}$ is the sign representation of $W' = W(\Phi')$. In type D we are thinking of the representation as truncated induction in type B followed by restriction to $W(D_n) \subset W(B_n)$ which is known to produce an irreducible representation as long as λ is not very even (we ignore the case where λ is very even since we never need to consider it in what follows).

Lemma 8. *When λ corresponds to a nilpotent orbit in the appropriate classical group, E_λ is the Springer representation of this orbit with the trivial local system.*

Proof. This is shown by following Lusztig's version of Shoji's algorithm (see chapter 13.3 of [Ca]). □

Lemma 9. *For two partitions λ, μ with the same X-collapse where $X = B, C,$ or D, we have $E_\lambda = E_\mu$ as representations of $W(X_n)$.*

Proof. We give the proof in type B, the other types being similar. Assume $\lambda \in \mathcal{P}(2n+1)$, but $\lambda \notin \mathcal{P}_B(2n+1)$. List all the even parts of λ in decreasing order as $\lambda_{e_1} \geq \lambda_{e_2} \geq \cdots \geq \lambda_{e_l}$. Then there exists an m such that $\lambda_{e_{2m-1}} > \lambda_{e_{2m}}$ since $\lambda \notin \mathcal{P}_B(2n+1)$. Let μ be the partition obtained from λ by replacing the part $\lambda_{e_{2m-1}}$ by $\lambda_{e_{2m-1}} - 1$ and the part $\lambda_{e_{2m}}$ by $\lambda_{e_{2m}} + 1$ and leaving all other parts of λ unchanged. This is the basic B-collapsing move (see [CM]) and it suffices to show that $E_\mu = E_\lambda$. The dual partition μ^* equals λ^* except that $\mu^*_{\lambda_{e_{2m}}+1} = \lambda^*_{\lambda_{e_{2m}}+1} + 1$ and $\mu^*_{\lambda_{e_{2m-1}}} = \lambda^*_{\lambda_{e_{2m-1}}} - 1$. Now write $\lambda^* = \alpha \cup \beta$ where α and β consist respectively of the odd and even parts of λ^*. Given a part λ^*_i of λ^*, recall that we are calling i the position of λ^*_i in λ^*. Now λ^*_i will occur as a part of either α or β. We refer to its position in whichever partition α or β it occurs in as its parity position. We use similar language for μ^*.

Assume first that $a = \lambda^*_{\lambda_{e_{2m}}+1}$ is even. Then it is possible to show that the parity position of $\lambda^*_{\lambda_{e_{2m}}+1}$ is odd and the parity position of $\mu^*_{\lambda_{e_{2m}}+1} = \lambda^*_{\lambda_{e_{2m}}+1} + 1$ is also odd. These parts will each contribute a root subsystem of type $B_{a/2}$ in the definition of E_λ or E_μ, respectively. On the other hand, if a is odd the parity position of these parts is both even and they will each contribute a root subsystem of type $D_{(a+1)/2}$ in the definition of E_λ or E_μ, respectively.

Since there are no even parts of λ between $\lambda_{e_{2m-1}}$ and $\lambda_{e_{2m}}$, we have $\lambda^*_{\lambda_{e_{2m}}+2j} = \lambda^*_{\lambda_{e_{2m}}+2j+1}$ for $j = 1, 2, \ldots, \frac{\lambda_{e_{2m-1}}-\lambda_{e_{2m}}}{2} - 1$. Since the same equalities hold for μ^*, we see that these parts (which come in pairs) contribute the same terms to E_λ and E_μ.

Finally consider $b = \lambda^*_{\lambda_{e_{2m-1}}}$. If b is even, then the parity position of both $\lambda^*_{\lambda_{e_{2m-1}}}$ and $\mu^*_{\lambda_{e_{2m-1}}}$ is even, so they contribute a term $D_{b/2}$ in the definition of E_λ or E_μ, respectively. And if b is odd, then they both have odd parity position and their contribution is $B_{(b-1)/2}$.

Hence a basic collapsing move does not affect the attached representation and the result is proved. □

Remark 10. The formulas for the $\tilde{b}$-values in the classical groups are a consequence of the previous propositions and the validity of property (B).

We now explain the bijection between $\mathcal{P}^{sp}_B(2n+1)$ and $\mathcal{P}^{sp}_C(2n)$ explicitly in terms of partitions (see [Sp], [KP]). Given $\lambda = [\lambda_1 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k] \in \mathcal{P}_B(2n+1)$, let $\lambda^- = [\lambda_1 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k - 1]$ and set $\lambda^C = \lambda^-_C$. Then $\lambda^C \in \mathcal{P}^{sp}_C(2n)$. Similarly given $\lambda = [\lambda_1 \geq \cdots \geq \lambda_k] \in \mathcal{P}_C(2n)$, let $\lambda^+ = [\lambda_1 + 1 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k]$ and $\lambda_+ = [\lambda_1 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k \geq 1]$ and set $\lambda^B = (\lambda^+)_B$. Then $\lambda^B \in \mathcal{P}^{sp}_B(2n+1)$ and moreover, $\lambda^B = (\lambda_+)^*_B)^*$. For $\lambda \in \mathcal{P}_B(2n+1)$, we have $(\lambda^C)^B = (\lambda^*_B)^*$; in particular, if λ is special, $(\lambda^C)^B = \lambda$. Similarly for $\lambda \in \mathcal{P}_C(2n)$, we have $(\lambda^B)^C = (\lambda^*_C)^*$; in particular, if λ is special, $(\lambda^B)^C = \lambda$.

In what follows we identify representations of $W(B_n)$ and $W(C_n)$ via the isomorphism of these two Coxeter groups which corresponds to interchanging long and short roots.

Lemma 11. *For $\lambda \in \mathcal{P}_C(2n)$ we have $E_{\lambda^*} = E_{(\lambda^B)^*}$, where the left side of the identity is computed in type C and the right in type B. For $\lambda \in \mathcal{P}_B(2n+1)$ we have $E_{\lambda^*} = E_{(\lambda^C)^*}$, where the left side of the identity is computed in type B and the right in type C.*

Proof. We prove the first isomorphism. We noted above that $(\lambda^B)^* = (\lambda_+)^*_B$ and so in type B, $E_{(\lambda^B)^*} = E_{(\lambda_+)^*}$ since by the previous proposition we can omit the B-collapse on the right side. To prove the desired identity we must study the odd and even parts of λ (in type C) and λ_+ (in type B). These partitions are the same except that the latter has an extra part equal to 1 at the end. Now because $\lambda \in \mathcal{P}_C(2n)$ the definition of the subsystem in equation (4) for E_{λ^*} in type C and for $E_{(\lambda_+)^*}$ in type B coincide (with the extra part in λ_+ playing no role at all).

We now prove the second isomorphism. The first isomorphism implies that $E_{(\lambda^C)^*} = E_{((\lambda^C)^B)^*}$, where the left side is in type C and the right in type B. Since $\lambda \in \mathcal{P}_B(2n+1)$, we have $(\lambda^C)^B = (\lambda^*_B)^*$ and thus $E_{((\lambda^C)^B)^*} = E_{(\lambda^*_B)} = E_{\lambda^*}$. The last equality holds since in type B we can omit the B-collapse. □

Theorem 12. *Our duality map $d : \mathcal{N}_{o,c} \to {}^L\mathcal{N}_o$ sends the pair (ν, η) to the orbit λ according to the following recipe:*

$$\lambda = \begin{cases} (\nu \cup \eta^C)^*_C & \mathfrak{g} \text{ type } B \\ (\nu \cup \eta^B)^*_B & \mathfrak{g} \text{ type } C \\ (\nu \cup (\eta^*_D)^*)^*_D & \mathfrak{g} \text{ type } D \end{cases} \tag{5}$$

Note that the case of ν equal to the empty partition corresponds to Lusztig-Spaltenstein duality. In type D if ν is non-empty, our assumptions about (ν, η) ensure that η is not very even.

Proof. We may choose $s \in Z_G(e)$ representing C so that $\mathfrak{l} = Z_{\mathfrak{g}}(s)$ has semisimple rank equal to the rank of $\mathfrak{g}$. Then $e \in \mathfrak{l}$ is specified by the pair of partitions (ν, η). Our first step is to compute the Springer representation $E_{d_{\mathfrak{l}}(e),1}$ of $W(s)$ associated to $d_{\mathfrak{l}}(e)$. The pair of partitions associated to $d_{\mathfrak{l}}(e)$ is

$$(6) \qquad \begin{aligned} &(\nu_D^*, \eta_B^*) \quad \mathfrak{g} \text{ type } B \\ &(\nu_C^*, \eta_C^*) \quad \mathfrak{g} \text{ type } C \\ &(\nu_D^*, \eta_D^*) \quad \mathfrak{g} \text{ type } D \end{aligned}$$

By lemmas 8 and 9, the associated Springer representation $E_{d_{\mathfrak{l}}(e),1}$ of $W(s)$ is $E_{\nu^*} \boxtimes E_{\eta^*}$. Consider this now as a representation of $W(D_k) \times W(C_{n-k})$ in type B, $W(C_k) \times W(B_{n-k})$ in type C, and $W(D_k) \times W(D_{n-k})$ in type D (there is no change in type D). Then by applying lemma 11 in types B and C and lemma 9 again in type D, this representation can be described as

$$(7) \qquad \begin{aligned} &E_{\nu^*} \boxtimes E_{(\eta^C)^*} \quad \mathfrak{g} \text{ type } B \\ &E_{\nu^*} \boxtimes E_{(\eta^B)^*} \quad \mathfrak{g} \text{ type } C \\ &E_{\nu^*} \boxtimes E_{(\eta^*)_D} \quad \mathfrak{g} \text{ type } D \end{aligned}$$

These representations possess property (B) as they possess property (B) in each simple component. Hence we can apply truncated induction up to $W({}^L\mathfrak{g})$. Then by transitivity of induction we claim that we arrive at the representation

$$(8) \qquad \begin{aligned} &E_{(\nu \cup \eta^C)^*} \quad \mathfrak{g} \text{ type } B \\ &E_{(\nu \cup \eta^B)^*} \quad \mathfrak{g} \text{ type } C \\ &E_{(\nu \cup (\eta_D^*)^*)^*} \quad \mathfrak{g} \text{ type } D \end{aligned}$$

where the first is a representation of $W(C_n)$, the second of $W(B_n)$, and the third of $W(D_n)$. In type D, we use the fact that $(\eta_D^*)^*$ belongs to $\mathcal{P}_C(2n-k)$. Therefore in all types if the multiplicity of i in ν is odd, then the multiplicity of i in η^C, η^B, or $(\eta_D^*)^*$, respectively, is even. Then the validity of equation (8) follows from the definition of E in equation (4).

The proof is completed by applying lemma 9 in ${}^L\mathfrak{g}$. $\square$

8. Bookends

Proposition 13. *Our duality map is surjective.*

Proof. We verified this case-by-case (Lusztig already did this in his work with his original map although the details are not recorded). In fact, we will try to exhibit canonical elements $(\mathcal{O}, C)$ of $\mathcal{N}_{o,c}$ which map bijectively to ${}^L\mathcal{N}_o^{sp}$. These are denoted by a star (*) in the tables for the exceptional groups and we now explain their construction in the classical groups.

Assume ${}^L\mathfrak{g}$ is of type X where $X = B, C, D$ and $\lambda \in \mathcal{P}_X(m)$ where m is even or odd depending on X. Consider λ^*. We ask whether λ^* belongs to $\mathcal{P}_B(m), \mathcal{P}_C(m)$, or $\mathcal{P}_C(m)$ depending on whether X is B, C, or D, respectively. In other words, we ask whether λ is special (note the funny situation in type D). If not, we may uniquely write $\lambda^* = \nu \cup \mu$

where ν is distinguished of type C, D, or D, and where μ belongs to $\mathcal{P}_B(m'), \mathcal{P}_C(m')$, or $\mathcal{P}_C(m')$, for some m', depending on whether X is B, C, or D.

We now show that μ has the property that $\mu^* \in \mathcal{P}_X(m')$. This is because $\lambda \in \mathcal{P}_X(m)$ and the process of forming μ^* amounts to taking λ and diminishing some of its parts; however, parts not congruent to ϵ will only be diminished by an even number, so the resulting partition remains of the same type as λ. Hence $\mu^* \in \mathcal{P}_X(m')$ for some m'. It follows that $\mu^* \in \mathcal{P}_X^{sp}(m')$ since μ belonged to $\mathcal{P}_B(m'), \mathcal{P}_C(m')$, or $\mathcal{P}_C(m')$ depending on X. It is also true that μ is itself special in types B and C.

We can now define η. In type B we set $\eta = \mu^C$; in type C we set $\eta = \mu^B$; and in type D we set $\eta = \mu_D$. Now because μ^* is special (and μ is special in types B and C), we have $\eta^B = (\mu^C)^B = \mu$ in type B; $\eta^C = (\mu^B)^C = \mu$ in type C; and $(\eta_D^*)^* = (((\mu^*)_D^*)_D^*)^* = (\mu^*)^* = \mu$ in type D. The second equality in type D follows since applying Lusztig-Spaltenstein duality twice (to μ^* in this case) is the identity on special orbits.

Thus in all types $d_{(\nu,\eta)} = (\nu \cup \mu)_X^* = (\lambda^*)_X^* = \lambda_X = \lambda$ where the last equality holds since $\lambda \in \mathcal{P}_X(m)$. We conclude in all types that (ν, η) has the property that $d_{(\nu,\eta)} = \lambda$. □

Remark 14. These canonical elements $(\mathcal{O}, C)$ that we have listed which surject onto ${}^L\mathcal{N}_o$ have the property that $\mathcal{O}$ is always special. In fact, all orbits $\mathcal{O}'$ of ${}^L\mathcal{N}_o$ in the same special piece are affiliated with the same special orbit $\mathcal{O}$ of $\mathfrak{g}$ and in fact $\mathcal{O} = d_{(\mathcal{O}',1)}$ (nilpotent orbits are in the same special piece exactly when their dual orbits are the same). Hence for each orbit $\mathcal{O}'$ in ${}^L\mathcal{N}_o$ we get a conjugacy class in $\bar{A}(\mathcal{O})$ where $\mathcal{O} = d_{(\mathcal{O}',1)}$. This should be the same conjugacy class that Lusztig attaches to orbits in [L3] under the identification $\bar{A}(\mathcal{O}) \cong \bar{A}(d_{(\mathcal{O},1)})$.

Proposition 15. *Let C, C' be conjugacy classes in $A(\mathcal{O})$ whose image in $\bar{A}(\mathcal{O})$ coincide, then $d_{(\mathcal{O},C)} = d_{(\mathcal{O},C')}$.*

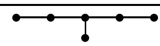
Proof. Again we checked this on a case-by-case basis. In the exceptional groups, this amounts to a quick glance at the tables which follow. In the classical groups, it requires attention to the computations in the proof of theorem 12, together with the explicit description of the canonical quotient. We omit the details. □

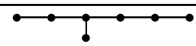
9. Duality in Exceptional Groups

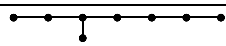
To compute the duality in the exceptional groups we used knowledge of Lusztig-Spaltenstein duality and the Springer correspondence (see [Ca]). Furthermore, we computed truncated induction by using the induce/restrict tables of Alvis [AL]. We have listed only those orbits with non-trivial component groups when G is of adjoint type since these are the only orbits for which we are saying something new. The stars(*) refer to the (putatively) canonical pair $(\mathcal{O}, C)$ which maps to a given orbit in the dual Lie algebra as is done in proposition 13 for the classical groups.

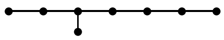
G_2				
	$(\mathfrak{l}, e)$	$\tilde{b}$	$\bar{A}(\mathcal{O})$	Dual
2 0	* $G_2(a_1)$	1	S_3	$G_2(a_1)$
	* $A_1 + \tilde{A}_1$	2		$\tilde{A}_1$
	* A_2	3		A_1

F_4				
	$(\mathfrak{l}, e)$	$\tilde{b}$	$\bar{A}(\mathcal{O})$	Dual
0 0 0 1	* $\tilde{A}_1$	1	S_2	$F_4(a_1)$
	$2A_1$	2		$F_4(a_2)$
2 0 0 0	* A_2	3	1	B_3
	$2A_1 + \tilde{A}_1$	3		B_3
2 0 0 1	B_2	4	S_2	$F_4(a_3)$
	A_3	6		B_2
1 0 1 0	$C_3(a_1)$	4	S_2	$F_4(a_3)$
	$A_1 + B_2$	5		$C_3(a_1)$
0 2 0 0	* $F_4(a_3)$	4	S_4	$F_4(a_3)$
	* $A_1 + C_3(a_1)$	5		$C_3(a_1)$
	* $A_2 + \tilde{A}_2$	6		$A_1 + \tilde{A}_2$
	* $B_4(a_2)$	6		B_2
	* $A_3 + \tilde{A}_1$	7		$A_2 + \tilde{A}_1$
0 2 0 2	* $F_4(a_2)$	10	1	$A_1 + \tilde{A}_1$
	$A_1 + C_3$	10		$A_1 + \tilde{A}_1$
2 2 0 2	* $F_4(a_1)$	13	S_2	$\tilde{A}_1$
	* B_4	16		A_1

E_6				
	$(\mathfrak{l}, e)$	$\tilde{b}$	$\bar{A}(\mathcal{O})$	Dual
$\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}$	$*\ A_2$ $*\ 4A_1$	3 4	S_2	$E_6(a_3)$ A_5
$\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}$	$*\ D_4(a_1)$ $*\ A_3+2A_1$ $*\ 3A_2$	7 8 9	S_3	$D_4(a_1)$ A_3+A_1 $2A_2+A_1$
$\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}$	$*\ E_6(a_3)$ $*\ A_5+A_1$	15 16	S_2	A_2 $3A_1$

E_7				
	$(\mathfrak{l}, e)$	$\tilde{b}$	$\bar{A}(\mathcal{O})$	Dual
2 0 0 0 0 0 0	* A_2 * $(4A_1)'$	3 4	S_2	$E_7(a_3)$ D_6
1 0 0 0 1 0 0	* A_2+A_1 $5A_1$	4 5	S_2	$E_6(a_1)$ $E_7(a_4)$
0 2 0 0 0 0 0	* $D_4(a_1)$ * $3A_2$ * $(A_3+2A_1)'$	7 9 8	S_3	$E_7(a_5)$ A_5+A_1 $D_6(a_2)$
0 1 0 0 0 1 1	* $D_4(a_1)+A_1$ * A_3+3A_1	8 9	S_2	$E_6(a_3)$ $(A_5)'$
0 0 1 0 1 0 0	* A_3+A_2 $D_4(a_1)+2A_1$	9 9	1	$D_5(a_1)+A_1$ $D_5(a_1)+A_1$
2 0 0 0 2 0 0	* A_4 * $2A_3$	10 12	S_2	$D_5(a_1)$ D_4+A_1
1 0 1 0 1 0 0	* A_4+A_1 A_1+2A_3	11 13	S_2	A_4+A_1 $A_3+A_2+A_1$
2 0 1 0 1 0 0	* $D_5(a_1)$ D_4+2A_1	13 14	S_2	A_4 A_3+A_2
0 2 0 0 2 0 0	* $E_6(a_3)$ * $(A_5+A_1)'$	15 16	S_2	$D_4(a_1)+A_1$ A_3+2A_1
0 0 2 0 0 2 0	* $E_7(a_5)$ * A_5+A_2 * $A_1+D_6(a_2)$	16 18 17	S_3	$D_4(a_1)$ $2A_2+A_1$ $(A_3+A_1)'$
2 0 2 0 0 2 0	* $E_7(a_4)$ $A_1+D_6(a_1)$	22 22	1	A_2+2A_1 A_2+2A_1
2 0 2 0 2 0 0	* $E_6(a_1)$ * A_7	25 28	S_2	A_2+A_1 $4A_1$
2 0 2 0 2 2 0	* $E_7(a_3)$ * A_1+D_6	30 31	S_2	A_2 $(3A_1)'$

E_8				
	$(\mathfrak{l}, e)$	$\tilde{b}$	$\bar{A}(\mathcal{O})$	Dual
0 0 0 0 0 0 2 0	$*\ A_2$ $*\ (4A_1)''$	3 4	S_2	$E_8(a_3)$ E_7
1 0 0 0 0 0 1 0	$*\ A_2 + A_1$ $5A_1$	4 5	S_2	$E_8(a_4)$ $E_8(b_4)$
2 0 0 0 0 0 0 0	$*\ 2A_2$ $*\ A_2 + 4A_1$	6 7	S_2	$E_8(a_5)$ D_7
0 0 0 0 0 2 0 0	$*\ D_4(a_1)$ $*\ 3A_2$ $*\ (A_3 + 2A_1)''$	7 9 8	S_3	$E_8(b_5)$ $E_6 + A_1$ $E_7(a_2)$
0 0 0 0 0 1 0 1	$*\ D_4(a_1) + A_1$ $3A_2 + A_1$ $A_3 + 3A_1$	8 10 9	S_3	$E_8(a_6)$ $E_8(b_6)$ $D_7(a_1)$
1 0 0 0 1 0 0 0	$*\ A_3 + A_2$ $D_4(a_1) + 2A_1$	9 9	1	$D_7(a_1)$ $D_7(a_1)$
0 0 0 0 0 0 0 2	$*\ D_4(a_1) + A_2$ $*\ A_3 + A_2 + 2A_1$	10 11	S_2	$E_8(b_6)$ A_7
2 0 0 0 0 0 2 0	$*\ A_4$ $*\ (2A_3)''$	10 12	S_2	$E_7(a_3)$ D_6
1 0 0 0 1 0 1 0	$*\ A_4 + A_1$ $A_1 + 2A_3$	11 13	S_2	$E_6(a_1) + A_1$ $D_5 + A_2$
0 0 1 0 0 0 1 0	$*\ A_4 + 2A_1$ $D_4(a_1) + A_3$	12 13	S_2	$D_7(a_2)$ $D_5 + A_2$
1 0 0 0 1 0 2 0	$*\ D_5(a_1)$ $D_4 + 2A_1$	13 14	S_2	$E_6(a_1)$ $E_7(a_4)$
0 0 0 0 0 0 2 2	$*\ D_4 + A_2$ $D_5(a_1) + 2A_1$	15 15	1	A_6 A_6
2 0 0 0 0 2 0 0	$*\ E_6(a_3)$ $*\ (A_5 + A_1)''$	15 16	S_2	$D_6(a_1)$ $D_5 + A_1$
0 1 0 0 0 1 0 1	$D_6(a_2)$ $D_4 + A_3$	16 18	S_2	$E_8(a_7)$ $D_6(a_2)$
1 0 0 1 0 1 0 0	$E_6(a_3) + A_1$ $A_5 + 2A_1$	16 17	S_2	$E_8(a_7)$ $E_7(a_5)$
0 0 1 0 1 0 0 0	$E_7(a_5)$ $A_5 + A_2$ $A_1 + D_6(a_2)$	16 18 17	S_3	$E_8(a_7)$ $E_6(a_3) + A_1$ $E_7(a_5)$

E_8				
	$(\mathfrak{l}, e)$	$\tilde{b}$	$\bar{A}(\mathcal{O})$	Dual
0 0 0 2 0 0 0 0	$* \ E_8(a_7)$ $* \ A_5+A_2+A_1$ $* \ 2A_4$ $* \ D_5(a_1)+A_3$ $* \ D_8(a_5)$ $* \ E_7(a_5)+A_1$ $* \ E_6(a_3)+A_2$	16 19 20 19 18 17 18	S_5	$E_8(a_7)$ A_5+A_1 A_4+A_3 $D_5(a_1)+A_2$ $D_6(a_2)$ $E_7(a_5)$ $E_6(a_3)+A_1$
0 1 0 0 0 1 2 1	$* \ D_6(a_1)$ $* \ D_5+2A_1$	21 22	S_2	$E_6(a_3)$ A_5
0 0 1 0 1 0 2 0	$* \ E_7(a_4)$ $A_1+D_6(a_1)$	22 22	1	$D_5(a_1)+A_1$ $D_5(a_1)+A_1$
0 0 0 2 0 0 2 0	$* \ D_5+A_2$ $E_7(a_4)+A_1$	23 23	1	A_4+A_2 A_4+A_2
1 0 1 0 1 0 1 0	$* \ D_7(a_2)$ $* \ D_5+A_3$	24 26	S_2	A_4+2A_1 $2A_3$
2 0 0 0 2 0 2 0	$* \ E_6(a_1)$ $* \ (A_7)''$	25 28	S_2	$D_5(a_1)$ D_4+A_1
1 0 1 0 1 0 2 0	$* \ E_6(a_1)+A_1$ A_7+A_1	26 29	S_2	A_4+A_1 $A_3+A_2+A_1$
0 0 2 0 0 0 2 0	$* \ E_8(b_6)$ $E_6(a_1)+A_2$ $* \ D_8(a_3)$	28 28 29	S_2	$D_4(a_1)+A_2$ $D_4(a_1)+A_2$ $A_3+A_2+A_1$
2 0 1 0 1 0 2 0	$* \ E_7(a_3)$ A_1+D_6	30 31	S_2	A_4 A_3+A_2
2 0 0 2 0 0 2 0	$* \ D_7(a_1)$ $E_7(a_3)+A_1$	31 31	1	A_3+A_2 A_3+A_2
0 0 2 0 0 2 0 0	$* \ E_8(a_6)$ $* \ A_8$ $* \ D_8(a_2)$	32 36 34	S_3	$D_4(a_1)+A_1$ $2A_2+2A_1$ A_3+2A_1
0 0 2 0 0 2 2 0	$* \ E_8(b_5)$ $* \ E_6+A_2$ $* \ E_7(a_2)+A_1$	37 39 38	S_3	$D_4(a_1)$ $2A_2+A_1$ A_3+A_1
2 0 2 0 0 2 0 0	$* \ E_8(a_5)$ $* \ D_8(a_1)$	42 43	S_2	$2A_2$ A_2+3A_1
2 0 2 0 0 2 2 0	$* \ E_8(b_4)$ $E_7(a_1)+A_1$	47 47	1	A_2+2A_1 A_2+2A_1
2 0 2 0 2 0 2 0	$* \ E_8(a_4)$ $* \ D_8$	52 56	S_2	A_2+A_1 $4A_1$
2 0 2 0 2 2 2 0	$* \ E_8(a_3)$ $* \ E_7+A_1$	63 64	S_2	A_2 $3A_1$

References

[AL] D. Alvis, *Induce/restrict matrices for exceptional Weyl groups*, available at `www.iusb.edu/~dalvis/`

[BV] D. Barbasch and D. Vogan, Jr., *Unipotent representations of complex semisimple groups*, Ann. of Math. (2), **121** (1985), no. 1, 41–110.

[BM] W. Borho and R. MacPherson, *Représentations des groupes de Weyl et homologie d'intersection pour les variétes nilpotentes*, C. R. Acad. Sci. Paris Sér. I Math., **292** (1981), no. 15, 707–710.

[Ca] R. W. Carter, *Finite Groups of Lie Type.* John Wiley and Sons: Chichester, 1985.

[CM] D. Collingwood and W. McGovern, *Nilpotent Orbits in Semisimple Lie algebras.* Van Nostrand Reinhold: New York, 1993.

[KP] H. Kraft and C. Procesi, *A special decomposition of the nilpotent cone of a classical Lie algebra*, Astérique **173-174** (1989), 271-279.

[L1] G. Lusztig, *A class of irreducible representations of a Weyl group I, II*, Indag. Math. **41** (1979), no. 3, 323–335; **44** (1982), no. 2, 219–226.

[L2] G. Lusztig, *Characters of reductive groups over a finite field.* Annals of Mathematics Studies, **107**. Princeton University Press, Princeton, N.J., 1984.

[L3] G. Lusztig, *Notes on unipotent classes*, Asian J. Math. **1** (1997), no. 1, 194–207.

[LuSp] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, J. London Math. Soc. (2), **19** (1979), no. 1, 41–52.

[So1] E. Sommers, *Nilpotent orbits and the affine flag manifold*, Ph.D. Thesis, M.I.T., 1997.

[So2] E. Sommers, *A generalization of the Bala-Carter theorem*, IMRN (Intl. Math. Res. Notices), 1998, 539-562.

[Sp] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel.* Lecture Notes in Mathematics, **946**. Springer-Verlag, Berlin-New York, 1982.

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